A GLOBAL EXISTENCE THEOREM FOR
LARGE EDDY SIMULATION TURBULENCE MODEL

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In this work we show the existence and uniqueness in Sobolev spaces of the solution of Large Eddy Simulation turbulence model for any time provided that initial data and external forces are regular and small enough. We also show that if external forces are time-periodic or time-independent, then the solution is time-periodic or time-independent.

1. Introduction

This paper deals with various aspects of Large Eddy Simulation (LES) turbulence model. This model has a more rigorous mathematical basis than the usual turbulence models which use Bousinnesq's assumption such as k–ε model.

As will be shown in Sec. 2, LES model averages Navier–Stokes equations over space using a Gaussian spatial filter and then approximates nonlinear terms using a Taylor expansion with respect to the filter width. The resulting equations for averaged quantities are like Navier–Stokes ones but with a nonlinear second-order term.

In Sec. 3 a global existence theorem for the solution of LES model is given provided that initial data and forces are small enough. We use a standard fixed point technique (Schauder's theorem) for nonlinear problems, introducing a continuous map from a convex compact set into itself. In order to deal with the high order nonlinear term introduced by LES model, we have to use norms on Sobolev spaces of high order and accept initial data only with a small enough $H^3(\Omega)$-norm. In this way we can show that the new function is still in the starting compact set. The solution found is then in $C^0([0,T]; H^3(\Omega)) \cap L^2(0,T; H^4(\Omega))$, with Ω an open, connected and bounded subset of $\mathbb{R}^3$ with regular boundary.

In Sec. 4 we provide a uniqueness theorem for small solutions and show the existence of periodic asympthotically stable solutions and therefore the existence of stationary solutions for external forces independent of time.

2. The Large Eddy Simulation Model

In this section we are going to show how LES model is built, starting from Navier–Stokes equations and using filtering techniques together with Fourier transform to spatially average velocity and pressure. We will not go in depth, which can be found in Catenkin et al.[2] and Aldama.[1]

Navier–Stokes equations for an incompressible, isotropic, Newtonian fluid are

$$
\begin{cases}
\partial_t u_i + \sum_{j=1}^{3} u_j \partial_j u_i = \nu \Delta u_i - \dfrac{1}{\rho}\partial_i P + f_i & \forall x \in \Omega \quad \forall t > 0 \quad i = 1,2,3 \\
\displaystyle\sum_{j=1}^{3} \partial_j u_j = 0 & \forall x \in \Omega \quad \forall t > 0,
\end{cases}
\tag{1}
$$

where ∂_t is the derivative with respect to time, u_i the velocity component, ∂_i the derivative with respect to x_i, ν the kinematic viscosity, Δ the Laplacian operator, i.e. $\sum_{i=1}^{3} \partial_i \partial_i$, ρ the density of the fluid, which is assumed to be constant, P its pressure and Ω an open, connected and bounded subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ with regular boundary.

LES turbulence model is now introduced, averaging velocities, pressure and equations on space and not on time as it is done in models which use Bousinnesq assumption like k–ε model. Following Catenkin et al.[2] we apply a homogeneous low pass filter to Eq. (1) using the following notation

$$
\overline{F}(x,t) = \int_{\mathbb{R}^n} G(x - x')F(x',t)dx' \,,
\tag{2}
$$

where G is the isotropic Gaussian low pass filter

$$
G(x) = \left(\frac{\gamma}{\pi}\right)^{\frac{3}{2}} \frac{1}{\lambda^3} \exp\left[-\gamma\left(\frac{x_j}{\lambda}\right)^2\right].
\tag{3}
$$

According to Aldama[1] the spatial filter width λ can be chosen in applications equal to twice the computational grid size and the free parameter γ equal to 6.

Each term of Navier–Stokes Eq. (1) is now filtered and, as in every classical approach to turbulence, we decompose velocities and pressure into a sum of resolved scale component $\overline{u_i}$, $\overline{P}$ and of subgrid scale component u_i', P'. Nonlinear terms are expanded as

$$
\overline{u_i u_j} = \overline{\overline{u_i}\,\overline{u_j}} + \overline{\overline{u_i} u_j'} + \overline{u_i' \overline{u_j}} + \overline{u_i' u_j'} \,.
\tag{4}
$$

Aldama[1] developed the following approximation technique for nonlinear terms: we apply convolution theorem to obtain

$$
\left\{\overline{\overline{u_i}\,\overline{u_j}}(k)\right\} = \left\{G(k)\right\}\left\{\overline{u_i}\,\overline{u_j}(k)\right\},
\tag{5}
$$

where $\{.\}$ denotes a Fourier transform and k is the wave number vector. Evaluating $\{G(k)\}$, expanding it in Taylor series with respect to λ and substituting it into (5) as in Ref. 2, gives

$$
\left\{\overline{\overline{u_i}\,\overline{u_j}}\right\} = \left\{\overline{u_i}\,\overline{u_j}\right\} - \frac{\lambda^2}{4\gamma}k_l k_l \left\{\overline{u_i}\,\overline{u_j}\right\} + O(\lambda^4).
\tag{6}
$$

Using the proprety of Fourier transform

$$
-k_l k_l \left\{\overline{u_i}\,\overline{u_j}\right\} = \left\{\partial_l \partial_l \overline{u_i}\,\overline{u_j}\right\}
$$

and taking the inverse Fourier transform, gives

$$
\overline{\overline{u_i}\,\overline{u_j}} = \overline{u_i}\,\overline{u_j} + \frac{\lambda^2}{4\gamma}\partial_l \partial_l \overline{u_i}\,\overline{u_j} + O(\lambda^4).
\tag{7}
$$

Applying the same method to the other right-hand side terms of (4) leads to

$$
\overline{u_i u_j} = \overline{u_i}\,\overline{u_j} + \frac{\lambda^2}{2\gamma}\sum_{l=1}^{3}\partial_l \overline{u_i}\,\partial_l \overline{u_j} + O(\lambda^4).
\tag{8}
$$

Indicating $\overline{u}$ and $\overline{P}$ with u and P, space filtered Navier–Stokes equations are now

$$
\begin{cases}
\partial_t u_i + \sum_{j=1}^{3} u_j \partial_j u_i = \nu \Delta u_i - \dfrac{1}{\rho}\partial_i P - \sum_{j,l=1}^{3}\partial_j\left[\dfrac{\lambda^2}{2\gamma}\partial_l u_i \partial_l u_j\right] + f_i + O(\lambda^4), \\
\displaystyle\sum_{j=1}^{3}\partial_j u_j = 0,
\end{cases}
\tag{9}
$$

which, together with boundary conditions $u_{|\partial\Omega} = 0$ and initial conditions $u_{|t=0} = u_0$ give the LES model. Clearly, the initial datum is required to satisfy

$$
\sum_{j=1}^{3}\partial_j u_{0j} = 0, \quad u_0{}_{|\partial\Omega} = 0.
\tag{10}
$$

3. Global Existence Theorem

In this section we are going to show that under compatibility conditions on initial and boundary data, if the initial values of velocity and external forces are small and regular enough, solution of LES model exists in $C^0([0,T]; H^3(\Omega)) \cap L^2(0,T; H^4(\Omega))$ for every $T > 0$.

Notation. From now on we will assume:

- every repeated index on the same side of an equality or inequality is summed;
- density ρ is equal to one;
- the expression DuD^2u means $\partial_j\left(\frac{\lambda^2}{2\gamma}\partial_l u_j \partial_l u_i\right) = \frac{\lambda^2}{2\gamma}\partial_l u_j \partial_l \partial_j u_i$. Every calculation will be made on the former expression only for simplicity of notation;
- c is an appropriate positive constant depending only on the domain Ω;
- f will be split as a sum of a gradient, which goes into the pressure term, and a divergence free field tangential to the boundary, which will still be indicated with f;
- $L^2(\Omega)$ is the Hilbert space of measurable functions whose square has a finite integral over Ω;
- $H^s(\Omega)$ with $s \geq 1$, s an integer, denotes the Hilbert space of functions in $L^2(\Omega)$ with distributional derivatives up to the sth order in $L^2(\Omega)$ and is called Sobolev space;
- $H^{\frac{1}{2}}(\partial\Omega)$ is the space of the traces on the boundary of functions in $H^1(\Omega)$. $H_0^1(\Omega)$ is the space of functions in $H^1(\Omega)$ with null trace on the boundary. When $H^s(\Omega)$ with $s \geq 1$ is referred to fluid velocity, it is always $H^s(\Omega) \cap H_0^1(\Omega)$;
- a Sobolev space with index div means that its elements are divergence free;
- $\partial_t u_0$ means

$$\partial_t u_0 := -u_0\nabla u_0 + \nu\Delta u_0 - Du_0 D^2 u_0 + f\big|_{t=0} - \nabla p_0. \tag{11}$$

Since initial value of pressure p_0 is not known, we have to get it by applying operator ∇ to differential Eq. (11)

$$\Delta p_0 = -\partial_i u_{j0}\partial_j u_{i0} - \frac{\lambda^2}{2\gamma}\partial_j\partial_l u_{i0}\partial_l\partial_i u_{j0} \quad \text{in } \Omega, \tag{12}$$

where boundary conditions are obtained multiplying Eq. (11) by the normal to the boundary n

$$\partial_n p_0 = \nu\Delta u_0 \cdot n - Du_0 D^2 u_0 \cdot n \quad \text{on } \partial\Omega.$$

In order to assure existence of p_0 we have to prove that

$$\int_\Omega -\partial_i u_{j0}\partial_j u_{i0} - \frac{\lambda^2}{2\gamma}\partial_j u_{i0}\partial_l\partial_i u_{j0}\, dx = \int_{\partial\Omega} \nu\Delta u_0 \cdot n - Du_0 D^2 u_0 \cdot n\, dx. \tag{13}$$

Adding the integral over Ω of $\nu\mathrm{div}\Delta u_0$, which is zero, to the left-hand side of (13) and integrating by parts, using the conditions (10) on initial datum, we easily get the right-hand side.

By using a typical technique for nonlinear problems the theorem will be proved showing the existence of a fixed point for a suitable continuous map. Namely, we build a continuous map which, starting from an initial function w, gives another function u, in the same convex and compact set and such that, if $u = w$, we have found the solution of our problem. In order to have every u in the same starting set as w, we will have to reduce the appropriate norm of initial datum and external forces. Since our fixed point map uses a differential equation to find the new u, we will prove that a solution exists and is unique using well-known results about linear nonstationary Navier–Stokes problem. Once divergence free velocity is found, pressure is recovered by means of orthogonality results.

Definition 1. (Compatibility conditions) We say that the initial datum $u_0 \in H^3(\Omega)$ satisfies compatibility conditions if u_0 and $\partial_t u_0$ have null trace on the boundary of Ω and div $u_0 = 0$.

These conditions are used to assure that LES model is also satisfied at initial time in order to estimate the $L^2(H^2)$-norm in terms of the $H^1(L^2)$-norm of the solution.

Theorem 1. (Existence) *There exists a $\delta_0 \in \mathbb{R}^+$ such that for every $\delta \in (0, \delta_0]$ if we assume that*

(a) *Ω is an open bounded connected set of $\mathbb{R}^3$ with regular boundary;*
(b) *initial condition u_0 satisfies compatibility conditions;*
(c) *$\|u_0\|_{H^3} \leq \delta^2$;*
(d) *$\|f\|_{L^2(H^2)} \leq \delta^2$ and $\|\partial_t f\|_{L^2(L^2)} \leq \delta^2$;*

then the solution of LES (9) exists in $C^0([0,T]; H^3(\Omega)) \cap L^2(0,T; H^4(\Omega))$ for fluid velocity and $C^0([0,T]; H^2(\Omega)) \cap L^2(0,T; H^3(\Omega))$ for pressure.

The proof requires several steps. Let us start by defining the following set

$$A = \Big\{ w : [0,T] \times \bar{\Omega} \to \mathbb{R}^3 \ \big| \ w|_{t=0} = u_0,$$
$$\|w\|_{L^\infty(H^3) \cap L^2(H^4)} \leq \delta \ \wedge \ \|\partial_t w\|_{L^\infty(H^1) \cap L^2(H^2)} \leq \delta \ \wedge \ \|\partial_t\partial_t w\|_{L^2(L^2)} \leq \delta \Big\},$$

with δ positive constant which will be defined later, and build the following map, which from w gives, after solving a differential problem, u:

$$\begin{cases} \partial_t u + \nabla p - \nu\Delta u = -w\nabla w - DwD^2 w + f \\ \text{div } u = 0 \\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0. \end{cases} \tag{14}$$

We define $F =: -w\nabla w - DwD^2 w + f$.

Let's introduce the orthogonal projection

$$P : L^2(\Omega) \to L^2_{\text{div}}(\Omega) = \big\{ u \in L^2 \ | \ u \cdot n = 0 \ \wedge \ \text{div } u = 0 \big\}.$$

We want to find $u \in L^2([0,T], H^1_{\mathrm{div}}(\Omega))$ such that

$$\begin{cases} \partial_t u - \nu P \Delta u = -P\big[w\nabla w + DwD^2 w\big] + f := F(t)\,, \\ u_{|t=0} = u_0\,. \end{cases} \qquad (15)$$

Due to the orthogonal decomposition $L^2(\Omega) = L^2_{\mathrm{div}}(\Omega) \oplus G$, where $G = \{v \in L^2(\Omega) \mid v = \nabla q \wedge q \in H^1(\Omega)\}$, this problem is equivalent to (14).

We need, at first, the following results:

Lemma 1. *If hypotheses of existence theorem are satisfied, then $\|F\|_{H^1(L^2)} \le c\delta^2$.*

Proof. We start with the $L^2(L^2)$ norm of F and use the fact that, since P is a projection, for every function ϕ one has $\|P\phi\|_{L^2} \le \|\phi\|_{L^2}$.

$$\int_0^T \|F(t)\|^2_{L^2}dt \le \int_0^T \|w\nabla w\|^2_{L^2}dt + \int_0^T \|DwD^2 w\|^2_{L^2}dt + \int_0^T \|f\|^2_{L^2}dt$$

$$\le c\|w\|^2_{L^2(H^2)}\|w\|^2_{L^\infty(H^1)} + c\|w\|^2_{L^2(H^3)}\|w\|^2_{L^\infty(H^2)} + \|f\|^2_{L^2(L^2)} \le c\delta^4\,.$$

Moreover, the time derivative $\partial_t F$ satisfies

$$\int_0^T \|\partial_t F(t)\|^2_{L^2}dt \le \int_0^T \|\partial_t(w\nabla w)\|^2_{L^2}dt$$

$$+ \int_0^T \|\partial_t(DwD^2 w)\|^2_{L^2}dt + \int_0^T \|\partial_t f\|^2_{L^2}dt$$

$$\le c\|\partial_t w\|^2_{L^\infty(L^2)}\|w\|^2_{L^2(H^3)} + c\|w\|^2_{L^\infty(H^3)}\|\partial_t w\|^2_{L^2(H^2)}$$

$$+ \|\partial_t f\|^2_{L^2(L^2)} \le c\delta^4\,. \qquad \square$$

Lemma 2. *If hypotheses of existence theorem hold, then $\|\partial_t u_0\|_{H^1} \le c\delta^2$.*

Proof. The true meaning of $\partial_t u_0$ is obviously the one given under Notation, therefore, from (12) we have

$$\|\nabla p_0\|_{H^1} \le c\|\partial_i u_{j0}\partial_j u_{i0}\|_{L^2} + c\|D^2 u_0 D^2 u_0\|_{L^2} + \|\tilde g\|_{H^1}$$

$$\le c\|u_0\|^2_{H^2} + c\|u_0\|^2_{H^3} + c\|u_0\|_{H^3} \le c\delta^2\,,$$

where $\tilde g = \nu\Delta u_0 \cdot \tilde n - Du_0 D^2 u_0 \cdot \tilde n$, i.e. the boundary value of Neumann problem extended to the whole domain Ω with $\tilde n$ the normal to the boundary extended to Ω.

Finally

$$\partial_t u_0 = -u_0\nabla u_0 + \nu\Delta u_0 - \nabla p_0 - Du_0 D^2 u_0 + f_{|t=0}$$

$$\|\partial_t u_0\|_{H^1} \le c\|u_0\|^2_{H^2} + c\|u_0\|_{H^3} + c\|\nabla p_0\|_{H^1} + c\|u_0\|^2_{H^3} + c\|f_{|t=0}\|_{H^1} \le c\delta^2\,. \qquad \square$$

We are now in a position to prove:

Proposition 1. *Under hypotheses of existence theorem, a solution of problem (15) exists in $C^0([0,t]; H^3(\Omega)) \cap L^2(0,T; H^4(\Omega))$ and $\|\partial_t u\|_{L^\infty(H^1)} + \|\partial_t\partial_t u\|_{L^2(L^2)} \le c\delta^2$.*

Proof. Since problems (14) and (15) are equivalent, using a result in Chap. 4, Corollary 2 of Ref. 4, if $F \in H^1(0,T; L^2(\Omega))$, if compatibility conditions hold and if $\partial_t u_0 \in H^1(\Omega)$ then u exists and is unique in $C^0([0,T]; H^3(\Omega)) \cap L^2(0,T; H^4(\Omega))$.

Now, deriving Eq. (14) with respect to time, multiplying it by $\partial_t\partial_t u$ and integrating over Ω, we get

$$\|\partial_t\partial_t u\|^2_{L^2} - \nu\int_\Omega P\Delta\partial_t u \cdot \partial_t\partial_t u\,dx = \int_\Omega \partial_t F \cdot \partial_t\partial_t u\,dx\,,$$

$$\|\partial_t\partial_t u\|^2_{L^2} + \frac{\nu}{2}\partial_t\|\partial_t\nabla u\|^2_{L^2} \le \varepsilon\|\partial_t\partial_t u\|^2_{L^2} + \frac{1}{4\varepsilon}\|\partial_t F\|^2_{L^2}\,.$$

Therefore, using Lemma 1, taking $\varepsilon = 1/2$ and integrating over $[0,T]$,

$$\|\partial_t\partial_t u\|^2_{L^2(L^2)} + c\nu\|\partial_t u\|^2_{L^\infty(H^1)} \le c\nu\|\partial_t u_0\|^2_{H^1} + \|\partial_t F\|^2_{L^2(L^2)}\,. \qquad (16)$$

Using compatibility conditions and thanks to Lemma 2 we can state that

$$\|\partial_t u\|_{L^\infty(H^1)} + \|\partial_t\partial_t u\|_{L^2(L^2)} \le c\delta^2\,. \qquad \square$$

Proposition 2. *If δ is small enough the set A is not empty.*

Proof. We take the $H^{1/2}(\partial\Omega)$ function

$$\psi := \Big(f_{|t=0} - Du_0 D^2 u_0 - u_0\nabla u_0 - \nabla p_0\Big)_{|\partial\Omega}$$

and extend it to a function Ψ in $C^0([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$, namely this function must satisfy $(\Psi_{|t=0})_{|\partial\Omega} = \psi$. This can be done thanks to Vol. 2, Chap. 4, Remark 3.3 of Ref. 5 (with $j = 0$, $m = 1$, $X = H^2$ and $Y = L^2$), which, in our case, states that the map which extends $H^1(\Omega)$ functions to $L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$ is surjective. We then consider the following heat problem

$$\begin{cases} \partial_t U - \nu\Delta U = \Psi\,, \\ U_{|t=0} = u_0 \quad U_{|\partial\Omega} = 0 \end{cases}$$

and observe that its solution belongs to $C^0([0,T]; H^3(\Omega)) \cap L^2(0,T; H^4(\Omega))$ since $u_0 \in H^3(\Omega)$, $\Psi \in C^0([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$ and compatibility conditions (those for the heat equation, not those in Definition 1) for $U_{|t=0}$ and $(\partial_t U)_{|t=0} = \Psi_{|t=0} + \nu\Delta u_0$ are automatically satisfied from our choice of Ψ. Moreover, due to the bound on u_0 and therefore on Ψ, choosing δ small enough, we have that $U \in A$. $\square$

Proposition 3. *The function u obtained from (14) belongs to A.*

Proof. Since we have proven that a solution u for problem (15) exists and that problem (15) is equivalent to (14), there exists a solution $u, \nabla p$ of (14). Observing that $\partial_t u - \nu \Delta u - F \in L^2(0,T; H^{-1}(\Omega))$ implies $\nabla p \in L^2(0,T; H^{-1}(\Omega))$, this solution satisfies the Stokes problem

$$\begin{cases} \nu \Delta u - \nabla p = \partial_t u + w\nabla w + DwD^2 w - f, \\ \operatorname{div} u = 0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{17}$$

If the term on the right-hand side of (17) is in $H^2(\Omega)$, we have (p. 40 of Ref. 4)

$$\|u\|_{H^3} + \|\nabla p\|_{H^1} \le c\|\partial_t u + w\nabla w + DwD^2 w - f\|_{H^1}, \tag{18}$$

$$\|u\|_{H^4} + \|\nabla p\|_{H^2} \le c\|\partial_t u + w\nabla w + w\nabla w + DwD^2 w - f\|_{H^2}, \tag{19}$$

which implies

$$\|u\|_{L^\infty(H^3)} + \|\nabla p\|_{L^\infty(H^1)} \le c\|\partial_t u\|_{L^\infty(H^1)} + c\|w\|_{L^\infty(H^2)}^2$$
$$+ c\|w\|_{L^\infty(H^3)}^2 + c\|f\|_{L^\infty(H^1)} \le c\delta^2.$$

To have the same estimate on norm $L^2(H^4)$, u and p can be seen as a solution of Stokes problem

$$\begin{cases} \nu\Delta\partial_t u - \nabla\partial_t p = \partial_t\partial_t u + \partial_t\big(w\nabla w\big) + \partial_t\big(DwD^2 w\big) - \partial_t f, \\ \operatorname{div}\partial_t u = 0, \\ \partial_t u|_{\partial\Omega} = 0. \end{cases}$$

Therefore we have

$$\|\partial_t u\|_{H^2} + \|\partial_t \nabla p\|_{L^2} \le c\Big\|\partial_t\partial_t u + \partial_t\big(w\nabla w\big) + \partial_t\big(DwD^2 w\big) - \partial_t f\Big\|_{L^2};$$

integrating this expression and (19) on $[0,T]$ we get

$$\|\partial_t u\|_{L^2(H^2)} \le c\|\partial_t\partial_t u\|_{L^2(L^2)} + c\|w\|_{L^2(H^4)}\|\partial_t w\|_{L^\infty(H^1)}$$
$$+ c\|w\|_{L^\infty(H^3)}\|\partial_t w\|_{L^2(H^2)} + c\|\partial_t f\|_{L^2(L^2)} \le c\delta^2.$$

Therefore

$$\|u\|_{L^2(H^4)} + \|\nabla p\|_{L^2(H^2)} \le c\|\partial_t u\|_{L^2(H^2)} + c\|w\|_{L^\infty(H^2)}\|w\|_{L^2(H^3)}$$
$$+ c\|w\|_{L^\infty(H^3)}\|w\|_{L^2(H^4)} + c\|f\|_{L^2(H^2)} \le c\delta^2.$$

Finally, if δ is small enough, we have shown that

$$\nabla p \in C^0([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)),$$

$$\|u\|_{L^\infty(H^3)} \le \delta, \qquad \|u\|_{L^2(H^4)} \le \delta,$$

$$\|\partial_t u\|_{L^\infty(H^1)} \le \delta, \qquad \|\partial_t u\|_{L^2(H^2)} \le \delta,$$

$$\|\partial_t\partial_t u\|_{L^2(L^2)} \le \delta.$$

This means that codomain of map (14) is A.   □

Proposition 4. (Compactness) *A is compact in $C^0([0,T]; H^2(\Omega))$.*

Proof. We observe that from inclusion $L^2(H^4) \cap H^1(H^2) \subset C^0(H^3)$ we can state that $A \subset C^0([0,T]; H^2(\Omega))$. From Ascoli-Arzelà theorem (Vol. 1, p. 142 of Ref. 3) A is relatively compact if and only if A is equicontinuous and for every $t \in [0,T]$ the set $A(t) = \{f(t) \in H^2(\Omega) \mid \forall f \in A\}$ is relatively compact in $H^2(\Omega)$.

Since in one dimension H^1 functions are Hölder functions, then A is immediately equicontinuous (Vol. 1, p. 142 of Ref. 3). Finally $A(t)$ is bounded in Hilbert space $H^3(\Omega)$ and therefore it is relatively weakly compact. Therefore, from $\{f_k\} \in A(t)$ we can extract $\{f_{k_h}\}$ which converges weakly in H^3 to f. From Rellich theorem H^3 is compact in H^2 and therefore $\{f_{k_h}\}$ converges strongly in H^2 to f. This means that A is relatively compact in $C^0([0,T]; H^2)$.

To prove that A is closed, we take a sequence $\phi_n \in A$ which converges in $C^0([0,T]; H^1(\Omega))$ to ϕ. We are going to use the facts that a bounded sequence in a Hilbert space has a subsequence which converges weakly and that a bounded sequence in $L^\infty(X)$, with X a Hilbert space, has a subsequence which is weakly* convergent; in both cases the norm of the limit function is not greater than the bound on the elements of the succession. Since $\partial_t \phi_n$ converges to $\partial_t \phi$ and $\partial_t\partial_t \phi_n$ to $\partial_t\partial_t \phi$, in the same way we have the bound on time derivatives of ϕ. Therefore $\phi \in A$.   □

Proposition 5. (Continuity of the map) *Under the same hypotheses of existence theorem, map (14) is continuous in $C^0([0,T], H^2(\Omega))$.*

Proof. We take a sequence w^k which converges to w in $C^0([0,T]; H^2(\Omega))$ and is bounded in $L^\infty(0,T; H^3(\Omega)) \cap L^2(0,T; H^4(\Omega))$. We want to show that the sequence u^k created from w^k by map (14) converges to u in $C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ and therefore, thanks to Lemma 4, it converges in $C^0(H^2)$ and this means that our map is continuous.

We define $v^k = w - w^k$ and $s^k = u - u6k$ and we substract differential equation for u^k from the one for u. The result is

$$\partial_t s^k - \nu\Delta s^k = -w\nabla w + w^k\nabla w^k - DwD^2 w + Dw^k D^2 w_k - \nabla p + \nabla p_k;$$

multiplying by s^k and integrating over Ω we get

$$\frac{1}{2}\partial_t\|s^k\|_{L^2}^2 + \nu\|s^k\|_{H^1}^2$$

$$\leq \int_\Omega \left|s^k v^k \nabla w\right|dx + \int_\Omega \left|s^k w^k \nabla v^k\right|dx$$

$$+ \int_\Omega \left|s^k Dv^k D^2 w\right|dx + \int_\Omega \left|s^k Dw^k D^2 v^k\right|dx$$

$$\leq c\|s^k\|_{L^2}\|v^k\|_{H^1}\|w\|_{H^2} + c\|s^k\|_{L^2}\|w^k\|_{H^2}\|v^k\|_{H^1}$$

$$+ c\|s^k\|_{L^2}\|v^k\|_{H^2}\|w\|_{H^3} + c\|s^k\|_{L^2}\|w^k\|_{H^3}\|v^k\|_{H^2}$$

$$\leq \varepsilon\|s^k\|_{L^2}^2 + \frac{c}{4\varepsilon}\delta^2\|v^k\|_{H^2}^2 .$$

Taking $\varepsilon = \nu/2$ and integrating on $[0,T]$, we have

$$\|s^k\|_{L^2(H^1)}^2 + \|s^k\|_{L^\infty(L^2)}^2 \leq c\delta^2\|v^k\|_{L^2(H^2)}^2 \to 0 .$$ □

We can now conclude the proof of Existence Theorem 1. In Proposition 5 we have proven that map (14) is continuous and therefore, since A is a nonempty, convex and compact set in the Banach space $C^0([0,T]; H^2(\Omega))$, using Schauder's theorem there exists a fixed point u of map (14). Clearly, this fixed point is a solution u of LES model, which is small in $C^0([0,T]; H^3(\Omega)) \cap L^2(0,T; H^4(\Omega))$. The corresponding pressure p satisfies $\nabla p \in C^0([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$.

4. Uniqueness and Periodic Solutions

Theorem 2. (Uniqueness) *Under the same hypotheses of existence theorem, for δ small enough a solution of LES model $u \in A$ is unique in A, while ∇p is unique.*

Proof. We define $s = u - v$, which satisfies

$$\begin{cases} \partial_t s - \nu\Delta s = -u\nabla u + v\nabla v - DuD^2u + DvD^2v - \nabla p_u + \nabla p_v , \\ \text{div } s = 0, \\ s|_{t=0} = 0, \quad s|_{\partial\Omega} = 0 . \end{cases} \quad (20)$$

If we multiply by s, integrate over Ω and use the fact that div $s = 0$ we get

$$\frac{1}{2}\partial_t\|s\|_{L^2}^2 + \nu\|s\|_{H^1}^2$$

$$\leq \int_\Omega \left|sj\partial_j u_i s_i\right|dx + c\int_\Omega \left(\left|\partial_l s_j\partial_j\partial_l u_i s_i\right| + \left|\partial_i v_j\partial_l s_i\partial_j s_i\right|\right)dx$$

$$\leq c\|s\|_{L^2}^2\|u\|_{H^3} + c\|s\|_{H^1}^2\left(\|u\|_{H^3} + \|v\|_{H^3}\right) .$$

Integrating on $[0,T]$ and remembering that u and v are small in $L^\infty(0,T; H^3(\Omega))$ we can obtain

$$\|s(t)\|_{L^2}^2 + \|s\|_{L^2(H^1)}^2 \leq c\delta\|s\|_{L^2(H^1)}^2 ,$$

$$\|s(t)\|_{L^2}^2 \leq 0 .$$

Uniqueness of ∇p follows from system (20). □

To have uniqueness of p a further condition on p must be imposed, such as

$$\int_\Omega p\, dx = 0 .$$

Theorem 3. (Stability) *Let v and u be two solutions in A with different initial values. If δ is small enough, the $L^2(\Omega)$ norm of their difference is controlled by the $L^2(\Omega)$ norm of the difference of their initial values and decreases exponentially with time.*

Proof. Let $s = v - u$. We have

$$\begin{cases} \partial_t s - \nu\Delta s = -v\nabla s - s\nabla u - DvD^2s - DsD^2u - \nabla(p_v - p_u) \\ \text{div } s = 0 \\ s|_{t=0} = v_0 - u_0 \quad s|_{\partial\Omega} = 0 . \end{cases} \quad (21)$$

We now multiply against s and integrate over Ω

$$\frac{1}{2}\partial_t\|s\|_{L^2}^2 + \nu\|s\|_{H^1}^2 \leq c\left[\|s\|_{L^2}^2\|u\|_{H^3} + \|s\|_{H^1}^2\|v\|_{H^3} + \|s\|_{H^1}^2\|u\|_{H^3}\right] \leq c\delta\|s\|_{H^1}^2 ,$$

having integrated by parts the terms $\int_\Omega v(\nabla s)s\, dx$ and $\int_\Omega Dv(D^2s)s\, dx$. Changing the value of c we easily have for δ small enough

$$\partial_t\|s\|_{L^2}^2 + c_0\|s\|_{L^2}^2 \leq 0 ;$$

$$\frac{\partial}{\partial t}\left(e^{c_0 t}\|s\|_{L^2}^2\right) \leq 0 ;$$

$$\|v(t) - u(t)\|_{L^2}^2 = \|s(t)\|_{L^2}^2 \leq e^{-c_0 t}\|s(0)\|_{L^2}^2 = e^{-c_0 t}\|v_0 - u_0\|_{L^2}^2 .$$ □

Theorem 4. (Periodic solution) *Let f be periodic of period $T > 0$ and let hypotheses of existence theorem be satisfied with δ^2 instead of δ. Then there exists a periodic solution of period T which is asymptotically stable and unique among any other solutions which satisfy existence theorem.*

Proof. If δ^2 is used instead of δ, our solution is smaller than δ^2 and the initial datum is smaller than δ^4.

We will follow here the approach of Serrin.[6] Let u be the solution of LES model with u_0 as initial value; let's define

$$\Phi_n(x) = u(nT, x) \quad \forall\, n \in \mathbb{N} ;$$

we want to show now that Φ_n is a Cauchy's sequence in $L^2(\Omega)$. Therefore we take two natural indices, n and m, with $m > n$ and define

$$w(t, x) = u(t + (m - n)T, x).$$

This means that w is a solution of LES model with initial value $u((m - n)T, x)$. Thanks to stability theorem we get

$$\|w(t) - u(t)\|_{L^2}^2 \leq e^{-c_0 t} \|u_0 - u((m - n)T, x)\|_{L^2} \leq 2\delta e^{-c_0 t}$$

which, taking $t = nT$, becomes

$$\|u(mt) - u(nT)\|_{L^2}^2 = \|w(nT) - u(nT)\|_{L^2}^2 \leq 2\delta e^{-c_0 nT}.$$

Therefore Φ_n is a Cauchy's sequence and $\Phi_n \to \Phi$ in $L^2(\Omega)$. The function Φ is also the weak limit in H^3, strong limit in H^s for every $s < 3$, in particular the uniform limit of Φ_n. Moreover, since the weak limit of a succession in H_0^1 remains in H_0^1, we have that compatibility conditions on $\partial_t \Phi$, which are

$$\partial_t \Phi = -\Phi \nabla \Phi + \nu \Delta \Phi - D\Phi D^2 \Phi + f\big|_{t=0} - \nabla p_\Phi,$$

where p_Φ satisfies

$$\Delta p_\Phi = -\partial_i u_{j0} \partial_j u_{i0} - \frac{\lambda^2}{2\gamma} \partial_j \partial_l u_{i0} \partial_l \partial_i u_{j0} \quad \text{in } \Omega$$

$$\partial_n p_\Phi = \nu \Delta u_0 \cdot n - Du_0 D^2 u_0 \cdot n \quad \text{on } \partial\Omega,$$

are satisfied for every Φ_n since they are solutions of LES model calculated at different times and $f\big|_{t=0} = f\big|_{t=nT}$, and therefore they are also satisfied for Φ. For the same reason, the divergence of Φ is zero and the $H^3(\Omega)$-norm of Φ is smaller than $c\delta^2$.

We now have to show that a solution v having Φ as initial value is periodic. Let's define

$$\bar{v}(t, x) = u(t + nT, x);$$

since f is periodic, $\bar{v}$ is a solution with initial value $\Phi_n(x)$ and therefore

$$\|v(t) - \bar{v}(t)\|_{L^2}^2 \leq e^{-c_0 t} \|\Phi - \Phi_n\|_{L^2}^2.$$

Taking $t = T$ we have

$$\|v(T) - \Phi_{n+1}\|_{L^2}^2 \leq e^{-c_0 T} \|\Phi - \Phi_n\|_{L^2}^2,$$

which becomes, when $n \to \infty$,

$$v(T) = \Phi = v(0).$$

Uniqueness follows from the fact that v is asymptotically stable.   □

Corollary 1. (Stationary solution) *Under the same hypothesis of periodic solution theorem, if f is time-independent, the asymptotically stable solution is constant in time.*

Proof. A constant function is periodic of period $1/n$, for every natural n. Therefore, once taken an initial value, v is unique and periodic for every $T \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, v is constant.   □

Acknowledgments

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